Proof of Form of $\boldsymbol{H}^{-1}$

Recall the definition of the matrix $\boldsymbol{H} \in \mathbb{R}^{L\times L}$ where $L = \frac{n(n-1)}{2}$.

$$\boldsymbol{H}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \langle \boldsymbol{w}_{\boldsymbol{\alpha}}, \boldsymbol{w}_{\boldsymbol{\beta}} \rangle = \begin{cases} 4 & \text{if } \alpha_1 = \beta_1 \,\&\, \alpha_2 = \beta_2 \\ 0 & \text{if } \alpha_1 \neq \beta_1 \,\&\, \alpha_1 \neq \beta_2 \,\&\, \alpha_2 \neq \beta_1 \,\&\, \alpha_2 \neq \beta_2 \\ 1 & \text{otherwise} \end{cases}$$

The main result of this note is an explicit form of $\boldsymbol{H}^{-1}$ which will be the focus of Lemma 2. In the proof of Lemma 2, a certain form of the basis $\boldsymbol{v}_{\boldsymbol{\alpha}}$ is used. The form is conjectured by inspection of the basis $\boldsymbol{v}_{\boldsymbol{\alpha}}$ for the case when n is small. We start by stating and proving this form.

Lemma 1. *Given an index (i, j) with $1 \le i < j \le n$, the matrix $\boldsymbol{v}_{i,j}$ has the following form.*

$$\boldsymbol{v}_{i,j} = \tilde{\boldsymbol{w}}_{i,j} + \boldsymbol{p}_{i,j} + \boldsymbol{q}_{i,j}$$

The matrices $\tilde{\boldsymbol{w}}_{ij}, \boldsymbol{p}_{i,j}$ and $\boldsymbol{q}_{ij}$ are respectively defined as follows.

$$\tilde{\boldsymbol{w}}_{i,j} = \frac{n-1}{n^2}\boldsymbol{e}_{i,i} + \frac{n-1}{n^2}\boldsymbol{e}_{j,j} - \frac{(n-1)^2+1}{2n^2}\boldsymbol{e}_{i,j} - \frac{(n-1)^2+1}{2n^2}\boldsymbol{e}_{j,i}$$

where $\boldsymbol{e}_{\alpha_1,\alpha_2}$ is a matrix of zeros except a 1 at the location (α_1, α_2). $\boldsymbol{p}_{ij}$ has the following form.

$$\boldsymbol{p}_{i,j} = \sum_{t,t\neq i,t\neq j} \frac{2n-4}{4n^2}\boldsymbol{e}_{i,t} + \sum_{s,s\neq i,s\neq j} \frac{2n-4}{4n^2}\boldsymbol{e}_{s,j}$$

$\boldsymbol{q}_{ij}$ is defined as follows.

$$\boldsymbol{q}_{i,j} = \sum_{(t,s),(t,s)\neq(i,j)} \boldsymbol{e}_{s,t}$$

where $(t, s) \neq (i, j)$ is defined as $\{t, s\} \cap \{i, j\} = \emptyset$.

Proof. To make the proposed form concrete, before proceeding with the proof, consider the following example.

Example 1: The form of $\boldsymbol{v}_{1,2}$ Consider the case where $n = 5$. Using the proposed form, the matrix $\boldsymbol{v}_{1,2}$ can be written as follows

$$\boldsymbol{v}_{1,2} = \tilde{\boldsymbol{w}}_{1,2} + \boldsymbol{p}_{1,2} + \boldsymbol{q}_{1,2}$$

where

$$\tilde{\boldsymbol{w}}_{1,2} = \frac{1}{25}\begin{pmatrix} 4 & -\frac{17}{2} & 0 & 0 & 0 \\ -\frac{17}{2} & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{p}_{1,2} = \frac{1}{25}\begin{pmatrix} 0 & 0 & 1.5 & 1.5 & 1.5 \\ 0 & 0 & 1.5 & 1.5 & 1.5 \\ 1.5 & 1.5 & 0 & 0 & 0 \\ 1.5 & 1.5 & 0 & 0 & 0 \\ 1.5 & 1.5 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\boldsymbol{q}_{1,2} = \frac{1}{25}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$

Therefore, $\boldsymbol{v}_{1,2}$ has the following explicit form

$$\boldsymbol{v}_{1,2} = \frac{1}{25}\begin{pmatrix} 4 & -\frac{17}{2} & 1.5 & 1.5 & 1.5 \\ -\frac{17}{2} & 4 & 1.5 & 1.5 & 1.5 \\ 1.5 & 1.5 & -1 & -1 & -1 \\ 1.5 & 1.5 & -1 & -1 & -1 \\ 1.5 & 1.5 & -1 & -1 & -1 \end{pmatrix}$$

As a first step simple check, consider $\langle \boldsymbol{v}_{1,2}, \boldsymbol{w}_{1,2} \rangle$ which results $\frac{1}{25}(8 + 17) = 1$ as desired.

The proof of Lemma 1 relies on showing that $\boldsymbol{v}_{i,j}$ is dual to $\boldsymbol{w}_{i,j}$. Since the dual basis is unique, establishing duality will conclude the proof. The result will be shown considering different cases.

Case 1: $\langle \boldsymbol{v}_{i,j}, \boldsymbol{w}_{i,j} \rangle$

$$\begin{aligned}\langle \boldsymbol{v}_{i,j}, \boldsymbol{w}_{i,j} \rangle = \langle \tilde{\boldsymbol{w}}_{i,j} + \boldsymbol{p}_{i,j} + \boldsymbol{q}_{i,j}, \boldsymbol{w}_{i,j} \rangle &= \langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{w}_{i,j} \rangle + \langle \boldsymbol{p}_{i,j}, \boldsymbol{w}_{i,j} \rangle + \langle \boldsymbol{q}_{i,j}, \boldsymbol{w}_{i,j} \rangle \\ &= \frac{2(n-1)}{n^2} + \frac{2(n-1)^2 + 2}{2n^2} + 0 + 0 = 1\end{aligned}$$

The second to last equality follows since $\langle \boldsymbol{p}_{i,j}, \boldsymbol{w}_{i,j} \rangle = 0$. and $\langle \boldsymbol{q}_{i,j}, \boldsymbol{w}_{i,j} \rangle = 0$.

Case 2: $\langle \boldsymbol{v}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle$ for $(i, j) \neq (\alpha, \beta)$

$$\begin{aligned}\langle \boldsymbol{v}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle = \langle \tilde{\boldsymbol{w}}_{i,j} + \boldsymbol{p}_{i,j} + \boldsymbol{q}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle &= \langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle + \langle \boldsymbol{p}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle + \langle \boldsymbol{q}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle \\ &= 0 + 0 + 0 = 0\end{aligned}$$

Case 3: $\langle \boldsymbol{v}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle$ for $\{i, j\} \cap \{\alpha, \beta\} \neq \emptyset$

A. $i = \alpha$ and $j \neq \beta$

$$\begin{aligned}\langle \boldsymbol{v}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle = \langle \tilde{\boldsymbol{w}}_{i,j} + \boldsymbol{p}_{i,j} + \boldsymbol{q}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle &= \langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle + \langle \boldsymbol{p}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle + \langle \boldsymbol{q}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle \\ &= 0 + 0 + 0 = 0\end{aligned}$$

B. $i \neq \alpha$ and $j = \beta$

$$\begin{aligned}\langle \boldsymbol{v}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle = \langle \tilde{\boldsymbol{w}}_{i,j} + \boldsymbol{p}_{i,j} + \boldsymbol{q}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle &= \langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle + \langle \boldsymbol{p}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle + \langle \boldsymbol{q}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle \\ &= 0 + 0 + 0 = 0\end{aligned}$$

C. $i = \beta$ and $j \neq \alpha$

$$\begin{aligned}\langle \boldsymbol{v}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle = \langle \tilde{\boldsymbol{w}}_{i,j} + \boldsymbol{p}_{i,j} + \boldsymbol{q}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle &= \langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle + \langle \boldsymbol{p}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle + \langle \boldsymbol{q}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle \\ &= 0 + 0 + 0 = 0\end{aligned}$$

D. $i \neq \beta$ and $j = \alpha$

$$\begin{aligned}\langle \boldsymbol{v}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle = \langle \tilde{\boldsymbol{w}}_{i,j} + \boldsymbol{p}_{i,j} + \boldsymbol{q}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle &= \langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle + \langle \boldsymbol{p}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle + \langle \boldsymbol{q}_{i,j}, \boldsymbol{w}_{\alpha,\beta} \rangle \\ &= 0 + 0 + 0 = 0\end{aligned}$$

With this, it can be concluded that the basis $\{\boldsymbol{v}_{i,j}\}$ is dual to $\{\boldsymbol{w}_{i,j}\}$ and the proposed form is established. □

The next Lemma uses the result of the above Lemma to derive an explicit form of the matrix $\boldsymbol{H}^{-1}$.

Lemma 2. *The inverse of the matrix $\boldsymbol{H}$, $\boldsymbol{H}^{-1}$, has the following explicit form*

$$\boldsymbol{H}^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \langle \boldsymbol{v}_{\boldsymbol{\alpha}}, \boldsymbol{v}_{\boldsymbol{\beta}} \rangle = \begin{cases} \dfrac{(n-1)^2 + 1}{2n^2} & \text{if } \alpha_1 = \beta_1 \,\&\, \alpha_2 = \beta_2 \\ \dfrac{1}{n^2} & \text{if } \alpha_1 \neq \beta_1 \,\&\, \alpha_1 \neq \beta_2 \,\&\, \alpha_2 \neq \beta_1 \,\&\, \alpha_2 \neq \beta_2 \\ \dfrac{4-2n}{4n^2} & \text{otherwise} \end{cases}$$

Proof. We consider three different cases.
Case 1: $\langle \boldsymbol{v}_{i,j}, \boldsymbol{v}_{i,j} \rangle$

$$\begin{aligned}\langle \boldsymbol{v}_{i,j}, \boldsymbol{v}_{i,j} \rangle &= \langle \tilde{\boldsymbol{w}}_{i,j} + \boldsymbol{p}_{i,j} + \boldsymbol{q}_{i,j}, \tilde{\boldsymbol{w}}_{i,j} + \boldsymbol{p}_{i,j} + \boldsymbol{q}_{i,j} \rangle \\ &= \langle \tilde{\boldsymbol{w}}_{i,j}, \tilde{\boldsymbol{w}}_{i,j} \rangle + \langle \boldsymbol{p}_{i,j}, \boldsymbol{p}_{i,j} \rangle + \langle \boldsymbol{q}_{i,j}, \boldsymbol{q}_{i,j} \rangle \\ &= 2\frac{(n-1)^2}{n^4} + 2\left[\frac{(n-1)^2+1}{2n^2}\right]^2 + \left(\frac{2n-4}{4n^2}\right)4(n-2) + \frac{1}{n^4}(n-2)^2 \\ &= \frac{(n-1)^2+1}{2n^2}\end{aligned}$$

The second equality uses the fact that $\langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{p}_{i,j} \rangle = 0$, $\langle \boldsymbol{p}_{i,j}, \boldsymbol{q}_{i,j} \rangle = 0$ and $\tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{q}_{i,j} = 0$. The third inequality simply uses the definition of $\tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{p}_{i,j}$ and $\boldsymbol{q}_{i,j}$. The last result follows after some algebraic manipulations.

Case 2: $\langle \boldsymbol{v}_{i,j}, \boldsymbol{v}_{\alpha,\beta} \rangle$ for $(i, j) \neq (\alpha,\beta)$, $\{i, j\} \cap \{\alpha, \beta\} = \varnothing$

$$\begin{aligned}\langle \boldsymbol{v}_{i,j}, \boldsymbol{v}_{\alpha,\beta} \rangle =& \langle \tilde{\boldsymbol{w}}_{i,j} + \boldsymbol{p}_{i,j} + \boldsymbol{q}_{i,j}, \tilde{\boldsymbol{w}}_{\alpha,\beta} + \boldsymbol{p}_{\alpha,\beta} + \boldsymbol{q}_{\alpha,\beta} \\ =& \langle \tilde{\boldsymbol{w}}_{i,j}, \tilde{\boldsymbol{w}}_{\alpha,\beta} \rangle + \langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{p}_{\alpha,\beta} \rangle + \langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle + \\ & \langle \boldsymbol{p}_{i,j}, \tilde{\boldsymbol{w}}_{\alpha,\beta} \rangle + \langle \boldsymbol{p}_{i,j}, \boldsymbol{p}_{\alpha,\beta} \rangle + \langle \boldsymbol{p}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle + \langle \boldsymbol{q}_{i,j}, \tilde{\boldsymbol{w}}_{\alpha,\beta} \rangle + \langle \boldsymbol{q}_{i,j}, \boldsymbol{p}_{\alpha,\beta} \rangle + \langle \boldsymbol{q}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle\end{aligned}$$

Each term will be evaluated separately.

1. $\langle \tilde{\boldsymbol{w}}_{i,j}, \tilde{\boldsymbol{w}}_{\alpha,\beta} \rangle = 0$ since for $(i, j) \neq (\alpha,\beta)$, $\tilde{\boldsymbol{w}}_{i,j}$ and $\tilde{\boldsymbol{w}}_{\alpha,\beta}$ have disjoint supports.

2. $\langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{p}_{\alpha,\beta} \rangle = 0$ since for $(i, j) \neq (\alpha,\beta)$, $\tilde{\boldsymbol{w}}_{i,j}$ and $\boldsymbol{p}_{\alpha,\beta}$ have disjoint supports.

3. Consider $\langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle$.

$$\begin{aligned}\langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle &= [\tilde{\boldsymbol{w}}_{i,j}]_{i,i}[\boldsymbol{q}_{\alpha,\beta}]_{i,i} + [\tilde{\boldsymbol{w}}_{i,j}]_{j,j}[\boldsymbol{q}_{\alpha,\beta}]_{j,j} + [\tilde{\boldsymbol{w}}_{i,j}]_{i,j}[\boldsymbol{q}_{\alpha,\beta}]_{i,j} + [\tilde{\boldsymbol{w}}_{i,j}]_{j,i}[\boldsymbol{q}_{\alpha,\beta}]_{j,i} \\ &= -2\frac{(n-1)}{n^4} + \frac{(n-1)^2+1}{n^4}\end{aligned}$$

4. $\langle \boldsymbol{p}_{i,j}, \tilde{\boldsymbol{w}}_{\alpha,\beta} \rangle = 0$ follows by a similar argument as 2.

5. Consider $\langle \boldsymbol{p}_{i,j}, \boldsymbol{p}_{\alpha,\beta} \rangle$.

$$\begin{aligned}\langle \boldsymbol{p}_{i,j}, \boldsymbol{p}_{\alpha,\beta} \rangle =& \sum_{s,s\neq i,s\neq j} [\boldsymbol{p}_{i,j}]_{i,s}[\boldsymbol{p}_{\alpha,\beta}]_{i,s} + \sum_{t,t\neq i,t\neq j} [\boldsymbol{p}_{i,j}]_{j,t}[\boldsymbol{p}_{\alpha,\beta}]_{j,t} + \sum_{s,s\neq i,s\neq j} [\boldsymbol{p}_{i,j}]_{s,i}[\boldsymbol{p}_{\alpha,\beta}]_{s,i} + \sum_{t,t\neq i,t\neq j} [\boldsymbol{p}_{i,j}]_{t,j}[\boldsymbol{p}_{\alpha,\beta}]_{t,j} \\ =& [\boldsymbol{p}_{i,j}]_{i,\alpha}[\boldsymbol{p}_{\alpha,\beta}]_{i,\alpha} + [\boldsymbol{p}_{i,j}]_{i,\beta}[\boldsymbol{p}_{\alpha,\beta}]_{i,\beta} + [\boldsymbol{p}_{i,j}]_{j,\alpha}[\boldsymbol{p}_{\alpha,\beta}]_{j,\alpha} + [\boldsymbol{p}_{i,j}]_{j,\beta}[\boldsymbol{p}_{\alpha,\beta}]_{j,\beta} + \\ & [\boldsymbol{p}_{i,j}]_{\alpha,i}[\boldsymbol{p}_{\alpha,\beta}]_{\alpha,i} + [\boldsymbol{p}_{i,j}]_{\beta,i}[\boldsymbol{p}_{\alpha,\beta}]_{\beta,i} + [\boldsymbol{p}_{i,j}]_{\alpha,j}[\boldsymbol{p}_{\alpha,\beta}]_{\alpha,j} + [\boldsymbol{p}_{i,j}]_{\beta,j}[\boldsymbol{p}_{\alpha,\beta}]_{\beta,j} \\ =& 8\left(\frac{2n-4}{4n^2}\right)^2 = \frac{2(n-2)^2}{n^4}\end{aligned}$$

6. Consider $\langle \boldsymbol{p}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle$.

$$\begin{aligned}\langle \boldsymbol{p}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle &= \sum_{t,t\neq i,t\neq j} [\boldsymbol{p}_{i,j}]_{i,t}[\boldsymbol{q}_{\alpha,\beta}]_{i,t} + \sum_{s,s\neq i,s\neq j} [\boldsymbol{p}_{i,j}]_{j,s}[\boldsymbol{q}_{\alpha,\beta}]_{j,s} + \sum_{t,t\neq i,t\neq j} [\boldsymbol{p}_{i,j}]_{t,i}[\boldsymbol{q}_{\alpha,\beta}]_{t,i} + \sum_{s,s\neq i,s\neq j} [\boldsymbol{p}_{i,j}]_{s,j}[\boldsymbol{q}_{\alpha,\beta}]_{s,j} \\ &= (4n-16)\frac{2n-4}{4n^2}\left(-\frac{1}{n^2}\right) = -\frac{2(n-4)(n-2)}{n^4}\end{aligned}$$

The first equality follows since it suffices to consider $\langle \boldsymbol{p}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle$ on the support of $\boldsymbol{p}_{i,j}$. The second and last equality result from the following analysis. Restricted to the support of $\boldsymbol{p}_{i,j}$, the matrix $\boldsymbol{q}_{\alpha,\beta}$ is non-zero except at the entries $[\boldsymbol{q}_{\alpha,\beta}]_{i,\alpha}, [\boldsymbol{q}_{\alpha,\beta}]_{i,\beta}, [\boldsymbol{q}_{\alpha,\beta}]_{\alpha,i}, [\boldsymbol{q}_{\alpha,\beta}]_{\beta,i}, [\boldsymbol{q}_{\alpha,\beta}]_{j,\alpha}, [\boldsymbol{q}_{\alpha,\beta}]_{j,\beta}, [\boldsymbol{q}_{\alpha,\beta}]_{\alpha,j}$ and $[\boldsymbol{q}_{\alpha,\beta}]_{\beta,j}$. Using this and the fact that $\boldsymbol{p}_{i,j}$ has $4(n-2)$ entries, $|\mathrm{supp}(\boldsymbol{p}_{i,j}) \cap \mathrm{supp}(\boldsymbol{q}_{\alpha,\beta})| = 4(n-2) - 8 = 4n - 16$. With this, the final form above follows.

7. $\langle \boldsymbol{q}_{i,j}, \tilde{\boldsymbol{w}}_{\alpha,\beta} \rangle = -2\frac{(n-1)}{n^4} + \frac{(n-1)^2+1}{n^4}$ follows by a similar argument as 3.

8. $\langle \boldsymbol{q}_{i,j}, \boldsymbol{p}_{\alpha,\beta} \rangle = -\dfrac{2(n-4)(n-2)}{n^4}$ follows by a similar argument as 6.

9. Consider $\langle \boldsymbol{q}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle$. Since $(i, j) \neq (\alpha, \beta)$, both $\boldsymbol{q}_{i,j}$ and $\boldsymbol{q}_{\alpha,\beta}$ have non-zero entry at (s, t) if and only if $s \neq i, s \neq j, s \neq \alpha, s \neq \beta, t \neq i, t \neq j, t \neq \alpha$ and $t \neq \beta$. Therefore, $|\mathrm{supp}(\boldsymbol{q}_{i,j}) \cap \mathrm{supp}(\boldsymbol{q}_{\alpha,\beta})| = (n-4)(n-4) = (n-4)^2$ given the $n-4$ choices for s and $n-4$ choices for t. $\langle \boldsymbol{q}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle$ can now be written as follows

$$\langle \boldsymbol{q}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle = \sum_{(s,t)\in \mathrm{supp}(\boldsymbol{q}_{i,j})\cap \mathrm{supp}(\boldsymbol{q}_{\alpha,\beta})} [\boldsymbol{q}_{i,j}]_{s,t}[\boldsymbol{q}_{\alpha,\beta}]_{s,t} = (n-4)^2 \left(-\frac{1}{n^2}\right)\left(-\frac{1}{n^2}\right) = \frac{(n-4)^2}{n^4}$$

Therefore, $\langle \boldsymbol{v}_{i,j}, \boldsymbol{v}_{\alpha,\beta} \rangle$ is the sum of the above terms.

$$\langle \boldsymbol{v}_{i,j}, \boldsymbol{v}_{\alpha,\beta} \rangle = -4\,\frac{n-1}{n^4} + 2\,\frac{(n-1)^2+1}{n^4} + \frac{2(n-2)^2}{n^4} - \frac{4(n-4)(n-2)}{n^4} + \frac{(n-4)^2}{n^4} = \frac{1}{n^2}$$

The last equality follows after some algebraic manipulations.

Case 3: $\langle \boldsymbol{v}_{i,j}, \boldsymbol{v}_{\alpha,\beta} \rangle$ for $\{i, j\} \cap \{\alpha, \beta\} \neq \emptyset$

A. $i = \alpha$ and $j \neq \beta$.

$$\begin{aligned}\langle \boldsymbol{v}_{i,j}, \boldsymbol{v}_{\alpha,\beta} \rangle =& \langle \tilde{\boldsymbol{w}}_{i,j} + \boldsymbol{p}_{i,j} + \boldsymbol{q}_{i,j}, \tilde{\boldsymbol{w}}_{\alpha,\beta} + \boldsymbol{p}_{\alpha,\beta} + \boldsymbol{q}_{\alpha,\beta} \rangle \\ =& \langle \tilde{\boldsymbol{w}}_{i,j}, \tilde{\boldsymbol{w}}_{\alpha,\beta} \rangle + \langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{p}_{\alpha,\beta} \rangle + \langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle + \\ & \langle \boldsymbol{p}_{i,j}, \tilde{\boldsymbol{w}}_{\alpha,\beta} \rangle + \langle \boldsymbol{p}_{i,j}, \boldsymbol{p}_{\alpha,\beta} \rangle + \langle \boldsymbol{p}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle + \langle \boldsymbol{q}_{i,j}, \tilde{\boldsymbol{w}}_{\alpha,\beta} \rangle + \langle \boldsymbol{q}_{i,j}, \boldsymbol{p}_{\alpha,\beta} \rangle + \langle \boldsymbol{q}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle\end{aligned}$$

Each term will be evaluated separately.

1. $\langle \tilde{\boldsymbol{w}}_{i,j}, \tilde{\boldsymbol{w}}_{\alpha,\beta} \rangle = [\tilde{\boldsymbol{w}}_{i,j}]_{i,i}[\tilde{\boldsymbol{w}}_{\alpha,\beta}]_{i,i} = \dfrac{(n-1)^2}{n^4}$.

2. Consider $\langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{p}_{\alpha,\beta} \rangle$.

$$\begin{aligned}\langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{p}_{\alpha,\beta} \rangle = [\tilde{\boldsymbol{w}}_{i,j}]_{i,j}[\boldsymbol{p}_{i,j}]_{i,j} + [\tilde{\boldsymbol{w}}_{i,j}]_{j,i}[\boldsymbol{p}_{i,j}]_{j,i} &= -\left(\frac{2(n-1)^2+2}{2n^2}\right)\left(\frac{2n-4}{4n^2}\right) \\ &= -\frac{(n-2)[(n-1)^2+1]}{2n^4}\end{aligned}$$

3. $\langle \tilde{\boldsymbol{w}}_{i,j}, \boldsymbol{q}_{\alpha,\beta} \rangle = [\tilde{\boldsymbol{w}}_{i,j}]_{j,j}[\boldsymbol{q}_{\alpha,\beta}]_{j,j} = \dfrac{n-1}{n^2}\left(-\dfrac{1}{n^2}\right) = -\dfrac{n-1}{n^4}$.

4. $\langle \boldsymbol{p}_{i,j}, \tilde{\boldsymbol{w}}_{\alpha,\beta} \rangle = -\dfrac{(n-2)[(n-1)^2+1]}{2n^4}$ follows by a similar argument as 2.

5. Consider $\langle \boldsymbol{p}_{i,j}, \boldsymbol{p}_{\alpha,\beta} \rangle$.

$$\begin{aligned}\langle \boldsymbol{p}_{i,j}, \boldsymbol{p}_{\alpha,\beta} \rangle =& \sum_{s,s\neq i,s\neq j} [\boldsymbol{p}_{i,j}]_{i,s}[\boldsymbol{p}_{\alpha,\beta}]_{i,s} + \sum_{t,t\neq i,t\neq j} [\boldsymbol{p}_{i,j}]_{j,t}[\boldsymbol{p}_{\alpha,\beta}]_{j,t} + \sum_{s,s\neq i,s\neq j} [\boldsymbol{p}_{i,j}]_{s,i}[\boldsymbol{p}_{\alpha,\beta}]_{s,i} + \sum_{t,t\neq i,t\neq j} [\boldsymbol{p}_{i,j}]_{t,j}[\boldsymbol{p}_{\alpha,\beta}]_{t,j} \\ =& \sum_{s,s\neq i,s\neq j} [\boldsymbol{p}_{i,j}]_{i,s}[\boldsymbol{p}_{\alpha,\beta}]_{i,s} + [\boldsymbol{p}_{i,j}]_{j,\beta}[\boldsymbol{p}_{\alpha,\beta}]_{j,\beta} + \sum_{s,s\neq i,s\neq j} [\boldsymbol{p}_{i,j}]_{s,i}[\boldsymbol{p}_{\alpha,\beta}]_{s,i} + [\boldsymbol{p}_{i,j}]_{t,\beta}[\boldsymbol{p}_{\alpha,\beta}]_{t,\beta} \\ =& (n-3)\left(\frac{2n-4}{4n^2}\right)^2 + \frac{2n-4}{4n^2} + (n-3)\left(\frac{2n-4}{4n^2}\right)^2 + \frac{2n-4}{4n^2} \\ =& 2(n-2)(n-3)\left(\frac{2n-4}{4n^2}\right)^2 = \frac{1}{2}\,\frac{(n-2)^3}{n^4}\end{aligned}$$

The second line follows since $[\boldsymbol{p}_{i,\beta}]_{j,t} = 0$ for all t except $t = \beta$ and $t = i$ and $[\boldsymbol{p}_{i,\beta}]_{t,j} = 0$ for all t except $t = \beta$ and $t = i$. The third line uses the fact that $[\boldsymbol{p}_{i,\beta}]_{i,s} \neq 0$ for all t except $t = i$ and $t = \beta$ and $[\boldsymbol{p}_{i,\beta}]_{s,i} \neq 0$ for all t except $t = i$ and $t = \beta$. The final equality results after some algebraic manipulations.

6. Consider $\langle \boldsymbol{p}_{i,j}\,, \boldsymbol{q}_{\alpha,\beta}\rangle$.

$$\begin{aligned}
\langle \boldsymbol{p}_{i,j}\,, \boldsymbol{q}_{\alpha,\beta}\rangle &= \sum_{t,t\neq i,t\neq j} [\boldsymbol{p}_{i,j}]_{i,t}[\boldsymbol{q}_{\alpha,\beta}]_{i,t} + \sum_{s,s\neq i,s\neq j} [\boldsymbol{p}_{i,j}]_{j,s}[\boldsymbol{q}_{\alpha,\beta}]_{j,s} + \sum_{t,t\neq i,t\neq j} [\boldsymbol{p}_{i,j}]_{t,i}[\boldsymbol{q}_{\alpha,\beta}]_{t,i} + \sum_{s,s\neq i,s\neq j} [\boldsymbol{p}_{i,j}]_{s,j}[\boldsymbol{q}_{\alpha,\beta}]_{s,j} \\
&= 0 + \sum_{s,s\neq i,s\neq j} [\boldsymbol{p}_{i,j}]_{j,s}[\boldsymbol{q}_{\alpha,\beta}]_{j,s} + 0 + \sum_{s,s\neq i,s\neq j} [\boldsymbol{p}_{i,j}]_{s,j}[\boldsymbol{q}_{\alpha,\beta}]_{s,j} \\
&= 2(n-3)\frac{2n-4}{4n^2}\left(-\frac{1}{n^2}\right) = -\frac{(n-3)(n-2)}{n^4}
\end{aligned}$$

The first equality follow since it suffices to consider $\langle \boldsymbol{p}_{i,j}\,, \boldsymbol{q}_{\alpha,\beta}\rangle$ on the support of $\boldsymbol{p}_{i,j}$. The second equality results since $[\boldsymbol{q}_{i,\beta}]_{i,t} = [\boldsymbol{q}_{i,\beta}]_{t,i} = 0$ for any t . The third and last equality result from the following analysis. Restricted to the support of $\boldsymbol{p}_{i,j}$, the matrix $\boldsymbol{q}_{i,\beta}$ is zero except at the entries $[\boldsymbol{q}_{i,\beta}]_{j,s}$ and $[\boldsymbol{q}_{i,\beta}]_{s,j}$ for all $j \neq \beta$. With this, $|\mathrm{supp}(\boldsymbol{p}_{i,j}) \cap \mathrm{supp}(\boldsymbol{q}_{\alpha,\beta})| = 2[(n-2)-1] = 2(n-3)$ and the final form follows.

7. $\langle \boldsymbol{q}_{i,j}\,, \tilde{\boldsymbol{w}}_{\alpha,\beta}\rangle = -\dfrac{n-1}{n^4}$ follows by a similar argument as 3.

8. $\langle \boldsymbol{q}_{i,j}\,, \boldsymbol{p}_{\alpha,\beta}\rangle = -\dfrac{(n-3)(n-2)}{n^4}$ follows by a similar argument as 6.

9. Consider $\langle \boldsymbol{q}_{i,j}\,, \boldsymbol{q}_{\alpha,\beta}\rangle$. Since $i = \alpha$ and $j \neq (\alpha, \beta)$, both $\boldsymbol{q}_{i,j}$ and $\boldsymbol{q}_{\alpha,\beta}$ have a non-zero entry at (s,t) if and only if $s \neq i, s \neq \alpha, s \neq \beta, t \neq i, t \neq j$, and $t \neq \beta$. Therefore, $|\mathrm{supp}(\boldsymbol{q}_{i,j}) \cap \mathrm{supp}(\boldsymbol{q}_{\alpha,\beta})| = (n-3)(n-3) = (n-3)^2$ given the $n-3$ choices for s and $n-3$ choices for t. $\langle \boldsymbol{q}_{i,j}\,, \boldsymbol{q}_{\alpha,\beta}\rangle$ can now be written as follows

$$\langle \boldsymbol{q}_{i,j}\,, \boldsymbol{q}_{\alpha,\beta}\rangle = \sum_{(s,t)\in \mathrm{supp}(\boldsymbol{q}_{i,j})\cap \mathrm{supp}(\boldsymbol{q}_{\alpha,\beta})} [\boldsymbol{q}_{i,j}]_{s,t}[\boldsymbol{q}_{\alpha,\beta}]_{s,t} = (n-3)^2\left(-\frac{1}{n^2}\right)\left(-\frac{1}{n^2}\right) = \frac{(n-3)^2}{n^4}$$

Therefore, $\langle \boldsymbol{v}_{i,j}\,, \boldsymbol{v}_{\alpha,\beta}\rangle$ is the sum of the above terms.

$$\begin{aligned}
\langle \boldsymbol{v}_{i,j}\,, \boldsymbol{v}_{\alpha,\beta}\rangle &= \frac{(n-1)^2}{n^4} + -\frac{(n-2)[(n-1)^2+1]}{n^4} - 2\frac{n-1}{n^4} + \frac{1}{2}\frac{(n-2)^3}{n^4} - 2\frac{(n-3)(n-2)}{n^4} + \frac{(n-3)^2}{n^4} \\
&= \frac{2-n}{2n^2}
\end{aligned}$$

The last equality follows after some algebraic manipulations.

Now the following three cases remain: $i \neq \alpha$ and $j = \beta$, $i = \beta$ and $j \neq \alpha$ and $i \neq \beta$ and $j = \alpha$. However, since $\boldsymbol{v}_{i,j}$ is symmetric the above argument can be adapted to these three cases by interchanging indices as appropriate. This concludes the proof.

□

Remark: A short proof of the form of $\boldsymbol{H}^{-1}$ might be plausible. The main technical challenge has been the locations of 1 and 0's in $\boldsymbol{H}$ which does not lend itself to simple analysis.