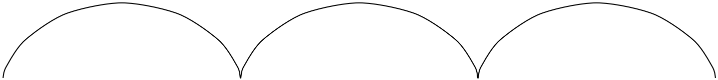
Hu-Washizu-Debeveque functional

Abiy Tasissa



December 4, 2012

OUTLINE

- Functional
- Euler's equation
- Vainberg's Theorem
- HWD functional
- Applications

THE PROBLEM OF SINGLE VARIABLE CALCULUS

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- Consider a function $f : R \to R$. We want to find the critical points.
- Compute f' and solve $f' = 0$
- Some problems in physics are minimization problems(minimum energy).
- We want to extend this notion but before doing so we need to understand functionals.

INTRODUCTION TO FUNCTIONALS

A functional is a mapping from a space of functions to a real number $f : F \to R$. What does this mean? Here are some examples.

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- Let $I = \int_a^b f(x)dx$. Then for well defined functions f, I is a functional.

CAN WE DO CALCULUS ON FUNCTIONALS?

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- This takes us into an area of mathematics called calculus of variations(functional analysis).
- The most important result in this area is due to Euler($1707 - 1783$).

FIRST VARIATION OF A FUNCTIONAL

- J stationary at u requires

$$\left.\frac{dJ(u+\epsilon\eta)}{d\epsilon}\right|_{\epsilon=0} = 0 = \underbrace{\langle DJ(u),\eta\rangle} = 0$$

for all admissible η

EULER'S EQUATION

- Let $F(x, y, z)$ be a C^2 function. Let us consider all the functions $y(x)$ which are continuously differentiable for $a \leq x \leq b$ and satisfy the boundary conditions

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- Euler showed that such a functional has to satisfy

$$F_y - \frac{d}{dx}F_{y'} = 0$$

EULER'S EQUATION APPLICATION

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$$0 - \frac{d}{dx}\frac{y'(x)}{\sqrt{1 + [y'(x)]^2}} = 0$$

$$y'(x) = \sqrt{\frac{c^2}{1 - c^2}}$$

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- Hence it is a straight line as desired. (To find explicitly apply boundary conditions).

EULER'S EQUATION EXTENDED

$$J(u) = \int_B F(x, u, \nabla u) - \int_{S_2} \phi(x, u) ds$$

The Euler equations are then given by:

$$\frac{\partial F}{\partial u_i} - \left(\frac{\partial F}{\partial u_{i,j}} \right)_{,j} = 0 \text{ in B}$$

$$\frac{\partial F}{\partial u_{i,j}} \cdot n_j = \frac{\partial \phi}{\partial u_i} \text{ on } S_2$$

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- That is let

$$\underbrace{\langle DJ(u), \eta \rangle}_{G(u,\eta)} = 0$$

$$G(u, \eta) = 0$$

When is $G(u, \eta)$ the first variation of a functional $J(u, \eta)$?

VAINBERG'S THEOREM(2)

- We apply Vainberg's reciprocity theorem. There is a functional $J(u)$ such that $\langle DJ(u), \eta\rangle = G(u,\eta)$ iff

$$\langle DG(u,\eta), \xi\rangle = \langle DG(u,\xi), \eta\rangle$$

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- If the condition is satisified

$$J(u) = \int_0^1 G(tu, u)\, dt$$

THE EQUATIONS OF LINEAR ELASTICITY

$$\sigma_{ij,j} + f_i = 0 \quad \text{in } B$$

$$\sigma_{ij} n_j = t_i \quad \text{on } S_2$$

$$\epsilon_{ij} = u_{(i,j)} \quad \text{in } B$$

$$u_i = \overline{u}_i \quad \text{on } S_1$$

$$\sigma_{ij}(\epsilon_{ij}) = \frac{\partial W}{\partial \epsilon_{ij}} \quad \text{in } B$$

STEP 1: INTEGRAL FORM (1)

- Weighted average sense. Corresponding to u, σ and ϵ, let the virtual parameters be η, α and β respectively.(This is quite the same step we took when we derived the principle of virtual work except now it is a three field formulation).

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- Weighted average sense. Corresponding to u, σ and ϵ, let the virtual parameters be η, α and β respectively.(This is quite the same step we took when we derived the principle of virtual work except now it is a three field formulation).
- $G\Bigg((u,\epsilon,\sigma),(\eta,\alpha,\beta)\Bigg)=$

$$\int_B \Bigg[\Bigg(-\sigma_{ij,j}+f_i\Bigg)\eta_i+\Bigg(\sigma_{ij}(\epsilon)-\frac{\partial W}{\partial \epsilon_{ij}}\Bigg)\alpha_{ij}+\Bigg(u(i,j)-\epsilon_{ij}\Bigg)\beta_{ij}\Bigg]dv+ \int_{S_2}\Bigg(\sigma_{ij}\eta_j-t_i\Bigg)\eta_i\,dS+\int_{S_1}\Bigg(\overline{u_i}-u_i\Bigg)\beta_{ij}\eta_j\,dS=0 \tag{1}$$

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- Note that each term represents work done by the virtual field.

STEP 1: INTEGRAL FORM (2)

- To simplify this, we use the following mathematical relationship:

$$\int \sigma_{ij,j}\,\eta_i\,dv = \int div(\eta_i\,\sigma_{ij})dV - \int \eta_{i,j}\,\sigma_{ij}\,dV$$

Apply divergence theorem

$$\sigma_{ij,j}\,\eta_i\,dV = \int_{S_1+S_2} (\sigma_{ij}\,\eta_i)n_j\,dS - \int \eta_{i,j}\,\sigma_{ij}\,dV$$

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- So we can rewrite (1) as $G\Big((u,\epsilon,\sigma),(\eta,\alpha,\beta)\Big)=$

$$\int_B \Big[-\sigma_{ij}\,\eta_{i,j}\,\eta_i - f_i\eta_i + \Big(\sigma_{ij}(\epsilon)-\sigma_{ij}\Big)\alpha_{ij} + \Big(u(i,j)-\epsilon_{ij}\Big)\beta_{ij}\Big]dv+$$

$$+\int_{S_1}\Big[\Big(u_i-\overline{u_i}\Big)\beta_{ij}+\sigma_{ij}\eta_i\Big]\,n_j\,dS - \int_{S_2}\bar{t}_i\,\eta_i\,dS = 0$$

STEP 2: CHECK RECIPROCITY

- Now we ask if the expression (2) can be derived from a functional. For that, we need to check the reciprocity condition.

$$\begin{aligned}&\left\langle DG\big((u,\epsilon,\sigma),(\eta,\alpha,\beta)\big),(\eta',\alpha',\beta')\right\rangle = \\ &\left\langle DG\big((u,\epsilon,\sigma),(\eta',\alpha',\beta')\big),(\eta,\alpha,\beta)\right\rangle\end{aligned} \qquad (3)$$

This can be shown to be true.

STEP 3: APPLY VAINBERG TO FIND FUNCTIONAL

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- $J(u, \epsilon, \sigma) =$

$$\int_0^1 G\Big((tu, t\epsilon, t\sigma), (u, \epsilon, \sigma)\Big)\, dt$$
$$= \int_0^1 \Big\{ \int_B \Big[- t\sigma_{ij}\, u_{(i,j)}\, u_i - f_i u_i + \Big(\sigma_{ij}(t\epsilon) - t\sigma_{ij}\Big)\epsilon_{ij} + \Big(tu_{(i,j)} - t\epsilon_{ij}\Big)\sigma_{ij}\Big] dV + \int_{S_1} \Big[t\Big(u_i - \overline{u_i}\Big)\sigma_{ij} + t\sigma_{ij}u_i\Big]\, n_j\, dS - \int_{S_2} \bar{t}_i\, u_i\, dS \Big\} dt \qquad (4)$$

STEP 4:HWD FUNCTIONAL

- Integrate (4) to find

$$J(u,\epsilon,\sigma) = \int_B \left[W(\epsilon) - f_i u_i + \sigma_{ij}(u_{i,j} - \epsilon_{ij}) \right] dV \qquad (5)$$

$$- \int_{S_1} \sigma_{ij} n_j \left(u_i - \overline{u_i} \right) dS - \int_{S_2} \bar{t}_i \, u_i dS \qquad (6)$$

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- This is the general functional in elasticity □

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- Remember

$$J(u) = \int_B F(x, u, \nabla u) - \int_{S_2} \phi(x, u) ds$$

- Hence

$$F(u, \epsilon, \sigma) = W(\epsilon) - f_i u_i + \sigma_{ij}(u_{i,j} - \epsilon_{ij})$$
$$\phi = \bar{t}_i u_i \text{ on } S_2$$
$$\phi = \sigma_{ij} n_j \left(u_i - \overline{u_i}\right) \text{ on } S_1$$

STEP 4: DOES THE FUNCTIONAL MAKE SENSE?(2)

- Now let's apply Euler's equations. Remember

$$\frac{\partial F}{\partial u_i} - \left(\frac{\partial F}{\partial u_{i,j}}\right)_{,j} = 0 \text{ in B}$$

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- On S_2
- $$\frac{\partial F}{\partial u_{i,j}} \cdot n_j = \frac{\partial \phi}{\partial u_i} \rightarrow \boxed{\sigma_{ij} n_j = t_i} \text{ stress-traction}$$

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$$\sigma_{ij} n_j \left(u_i - \overline{u_i} \right) = 0 \rightarrow \boxed{u_i = \overline{u_i}} \text{ Displacement Boundary}$$

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- Now let's see a bit of the history and where this can be applied.

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- Hence I have used the name Hu-Washizu-De Veubeke functional.

APPLICATIONS / / LOCKING PROBLEM

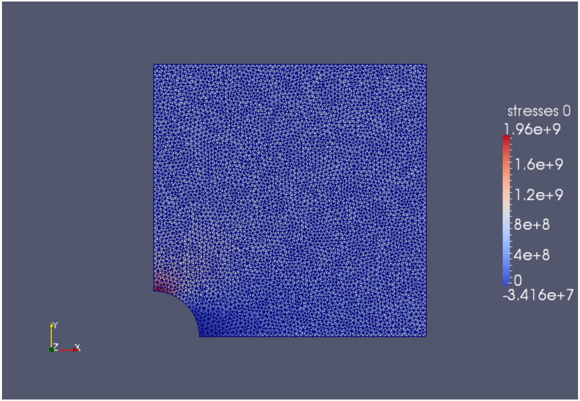

Figure: $\sigma_{11}, \nu = 0.3$

APPLICATIONS / / LOCKING PROBLEM 2

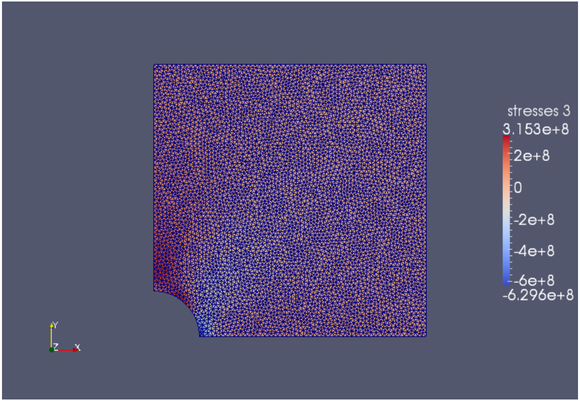

Figure: σ_{33}, $\nu = 0.3$

APPLICATIONS / /LOCKING PROBLEM

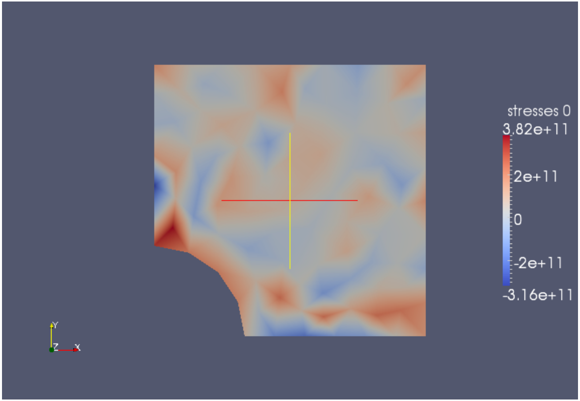

Figure: $\sigma_{11}, \nu = 0.4999$

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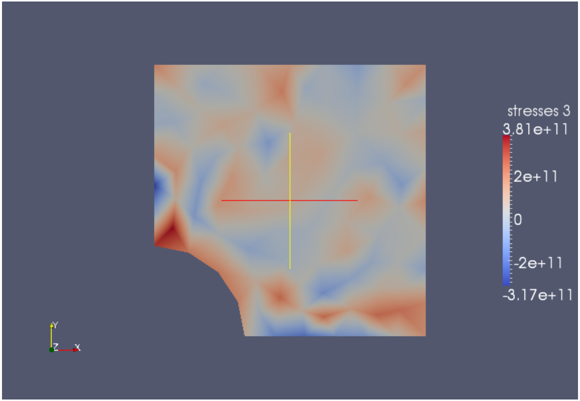

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LOCKING PROBLEM 2

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- In principle, one could go high order and solve these but it comes at a price of computational complexity.

CONCLUSION

- This approach we took is not limited to elasticity and it shouldn't be. The mathematics can be applied to a lot of other areas.

REFERENCES

- Gelfand, Fomin, *Calculus of Variations*, Prentice-Hall, 1963.
- Fraeijs de Veubeke, B. M., *Variational principles and the patch test* Int. J. Numer. Meth. Engrg, 8, 783-801, 1974.
- Hu, H.-C., *On some variational methods on the theory of elasticity and the theory of plasticity* Scientia Sinica, 4, 33-54, 1955

REFERENCES

- M.M. Vainberg *Variational Methods for the Study of Nonlinear Operators*, Holden-day, 1964.
- Washizu, K., *On the variational principles of elasticity and plasticity, Aeroelastic and Structures* Research Laboratory, Technical Report 25-18, MIT, Cambridge, 1955.

THANK YOU for your attention!